

DIFFERENTIATION RULES

General Formulas

Assume u and v are differentiable functions of x .

Constant: $\frac{d}{dx}(c) = 0$

Sum: $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Difference: $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$

Constant Multiple: $\frac{d}{dx}(cu) = c \frac{du}{dx}$

Product: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Power: $\frac{d}{dx}x^n = nx^{n-1}$

Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \quad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Exponential and Logarithmic Functions

$$\frac{d}{dx}e^x = e^x \quad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}a^x = a^x \ln a \quad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2} \quad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \quad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \quad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \quad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \quad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}}$$

Parametric Equations

If $x = f(t)$ and $y = g(t)$ are differentiable, then

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

A BRIEF TABLE OF INTEGRALS

1. $\int u dv = uv - \int v du$
2. $\int a^u du = \frac{a^u}{\ln a} + C, \quad a \neq 1, \quad a > 0$
3. $\int \cos u du = \sin u + C$
4. $\int \sin u du = -\cos u + C$
5. $\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$
6. $\int (ax + b)^{-1} dx = \frac{1}{a} \ln |ax + b| + C$
7. $\int x(ax + b)^n dx = \frac{(ax + b)^{n+1}}{a^2} \left[\frac{ax + b}{n+2} - \frac{b}{n+1} \right] + C, \quad n \neq -1, -2$
8. $\int x(ax + b)^{-1} dx = \frac{x}{a} - \frac{b}{a^2} \ln |ax + b| + C$
9. $\int x(ax + b)^{-2} dx = \frac{1}{a^2} \left[\ln |ax + b| + \frac{b}{ax + b} \right] + C$
10. $\int \frac{dx}{x(ax + b)} = \frac{1}{b} \ln \left| \frac{x}{ax + b} \right| + C$
11. $\int (\sqrt{ax + b})^n dx = \frac{2}{a} \frac{(\sqrt{ax + b})^{n+2}}{n+2} + C, \quad n \neq -2$
12. $\int \frac{\sqrt{ax + b}}{x} dx = 2\sqrt{ax + b} + b \int \frac{dx}{x\sqrt{ax + b}}$
13. (a) $\int \frac{dx}{x\sqrt{ax - b}} = \frac{2}{\sqrt{b}} \tan^{-1} \sqrt{\frac{ax - b}{b}} + C$ (b) $\int \frac{dx}{x\sqrt{ax + b}} = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax + b} - \sqrt{b}}{\sqrt{ax + b} + \sqrt{b}} \right| + C$
14. $\int \frac{\sqrt{ax + b}}{x^2} dx = -\frac{\sqrt{ax + b}}{x} + \frac{a}{2} \int \frac{dx}{x\sqrt{ax + b}} + C$
15. $\int \frac{dx}{x^2\sqrt{ax + b}} = -\frac{\sqrt{ax + b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax + b}} + C$
16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$
17. $\int \frac{dx}{(a^2 + x^2)^2} = \frac{x}{2a^2(a^2 + x^2)} + \frac{1}{2a^3} \tan^{-1} \frac{x}{a} + C$
18. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{x + a}{x - a} \right| + C$
19. $\int \frac{dx}{(a^2 - x^2)^2} = \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{4a^3} \ln \left| \frac{x + a}{x - a} \right| + C$
20. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a} + C = \ln(x + \sqrt{a^2 + x^2}) + C$
21. $\int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C$
22. $\int x^2 \sqrt{a^2 + x^2} dx = \frac{x}{8} (a^2 + 2x^2) \sqrt{a^2 + x^2} - \frac{a^4}{8} \ln(x + \sqrt{a^2 + x^2}) + C$

23. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right| + C$
24. $\int \frac{\sqrt{a^2 + x^2}}{x^2} dx = \ln(x + \sqrt{a^2 + x^2}) - \frac{\sqrt{a^2 + x^2}}{x} + C$
25. $\int \frac{x^2}{\sqrt{a^2 + x^2}} dx = -\frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + \frac{x\sqrt{a^2 + x^2}}{2} + C$
26. $\int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right| + C$
27. $\int \frac{dx}{x^2\sqrt{a^2 + x^2}} = -\frac{\sqrt{a^2 + x^2}}{a^2x} + C$
28. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$
29. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$
30. $\int x^2 \sqrt{a^2 - x^2} dx = \frac{a^4}{8} \sin^{-1} \frac{x}{a} - \frac{1}{8} x \sqrt{a^2 - x^2} (a^2 - 2x^2) + C$
31. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + C$
32. $\int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\sin^{-1} \frac{x}{a} - \frac{\sqrt{a^2 - x^2}}{x} + C$
33. $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} - \frac{1}{2} x \sqrt{a^2 - x^2} + C$
34. $\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + C$
35. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2x} + C$
36. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C$
37. $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + C$
 $= \ln|x + \sqrt{x^2 - a^2}| + C$
38. $\int (\sqrt{x^2 - a^2})^n dx = \frac{x(\sqrt{x^2 - a^2})^n}{n+1} - \frac{na^2}{n+1} \int (\sqrt{x^2 - a^2})^{n-2} dx, \quad n \neq -1$
39. $\int \frac{dx}{(\sqrt{x^2 - a^2})^n} = \frac{x(\sqrt{x^2 - a^2})^{2-n}}{(2-n)a^2} - \frac{n-3}{(n-2)a^2} \int \frac{dx}{(\sqrt{x^2 - a^2})^{n-2}}, \quad n \neq 2$
40. $\int x(\sqrt{x^2 - a^2})^n dx = \frac{(\sqrt{x^2 - a^2})^{n+2}}{n+2} + C, \quad n \neq -2$
41. $\int x^2 \sqrt{x^2 - a^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{x^2 - a^2} - \frac{a^4}{8} \ln|x + \sqrt{x^2 - a^2}| + C$
42. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \sec^{-1} \left| \frac{x}{a} \right| + C$
43. $\int \frac{\sqrt{x^2 - a^2}}{x^2} dx = \ln|x + \sqrt{x^2 - a^2}| - \frac{\sqrt{x^2 - a^2}}{x} + C$
44. $\int \frac{x^2}{\sqrt{x^2 - a^2}} dx = \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + \frac{x}{2} \sqrt{x^2 - a^2} + C$
45. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C = \frac{1}{a} \cos^{-1} \left| \frac{a}{x} \right| + C$
46. $\int \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2x} + C$

$$47. \int \frac{dx}{\sqrt{2ax - x^2}} = \sin^{-1} \left(\frac{x-a}{a} \right) + C$$

$$48. \int \sqrt{2ax - x^2} dx = \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + C$$

$$49. \int (\sqrt{2ax - x^2})^n dx = \frac{(x-a)(\sqrt{2ax - x^2})^n}{n+1} + \frac{na^2}{n+1} \int (\sqrt{2ax - x^2})^{n-2} dx$$

$$50. \int \frac{dx}{(\sqrt{2ax - x^2})^n} = \frac{(x-a)(\sqrt{2ax - x^2})^{2-n}}{(n-2)a^2} + \frac{n-3}{(n-2)a^2} \int \frac{dx}{(\sqrt{2ax - x^2})^{n-2}}$$

$$51. \int x\sqrt{2ax - x^2} dx = \frac{(x+a)(2x-3a)\sqrt{2ax - x^2}}{6} + \frac{a^3}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + C$$

$$52. \int \frac{\sqrt{2ax - x^2}}{x} dx = \sqrt{2ax - x^2} + a \sin^{-1} \left(\frac{x-a}{a} \right) + C$$

$$53. \int \frac{\sqrt{2ax - x^2}}{x^2} dx = -2\sqrt{\frac{2a-x}{x}} - \sin^{-1} \left(\frac{x-a}{a} \right) + C$$

$$54. \int \frac{x dx}{\sqrt{2ax - x^2}} = a \sin^{-1} \left(\frac{x-a}{a} \right) - \sqrt{2ax - x^2} + C$$

$$55. \int \frac{dx}{x\sqrt{2ax - x^2}} = -\frac{1}{a} \sqrt{\frac{2a-x}{x}} + C$$

$$56. \int \sin ax dx = -\frac{1}{a} \cos ax + C$$

$$57. \int \cos ax dx = \frac{1}{a} \sin ax + C$$

$$58. \int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$$

$$59. \int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$$

$$60. \int \sin^n ax dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int \sin^{n-2} ax dx$$

$$61. \int \cos^n ax dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax dx$$

$$62. \text{(a)} \int \sin ax \cos bx dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + C, \quad a^2 \neq b^2$$

$$\text{(b)} \int \sin ax \sin bx dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$$

$$\text{(c)} \int \cos ax \cos bx dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$$

$$63. \int \sin ax \cos ax dx = -\frac{\cos 2ax}{4a} + C$$

$$64. \int \sin^n ax \cos ax dx = \frac{\sin^{n+1} ax}{(n+1)a} + C, \quad n \neq -1$$

$$65. \int \frac{\cos ax}{\sin ax} dx = \frac{1}{a} \ln |\sin ax| + C$$

$$66. \int \cos^n ax \sin ax dx = -\frac{\cos^{n+1} ax}{(n+1)a} + C, \quad n \neq -1$$

$$67. \int \frac{\sin ax}{\cos ax} dx = -\frac{1}{a} \ln |\cos ax| + C$$

68. $\int \sin^n ax \cos^m ax \, dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(m+n)} + \frac{n-1}{m+n} \int \sin^{n-2} ax \cos^m ax \, dx, \quad n \neq -m \quad (\text{reduces } \sin^n ax)$
69. $\int \sin^n ax \cos^m ax \, dx = \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(m+n)} + \frac{m-1}{m+n} \int \sin^n ax \cos^{m-2} ax \, dx, \quad m \neq -n \quad (\text{reduces } \cos^m ax)$
70. $\int \frac{dx}{b+c \sin ax} = \frac{-2}{a\sqrt{b^2-c^2}} \tan^{-1} \left[\sqrt{\frac{b-c}{b+c}} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) \right] + C, \quad b^2 > c^2$
71. $\int \frac{dx}{b+c \sin ax} = \frac{-1}{a\sqrt{c^2-b^2}} \ln \left| \frac{c+b \sin ax + \sqrt{c^2-b^2} \cos ax}{b+c \sin ax} \right| + C, \quad b^2 < c^2$
72. $\int \frac{dx}{1+\sin ax} = -\frac{1}{a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) + C$
73. $\int \frac{dx}{1-\sin ax} = \frac{1}{a} \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) + C$
74. $\int \frac{dx}{b+c \cos ax} = \frac{2}{a\sqrt{b^2-c^2}} \tan^{-1} \left[\sqrt{\frac{b-c}{b+c}} \tan \frac{ax}{2} \right] + C, \quad b^2 > c^2$
75. $\int \frac{dx}{b+c \cos ax} = \frac{1}{a\sqrt{c^2-b^2}} \ln \left| \frac{c+b \cos ax + \sqrt{c^2-b^2} \sin ax}{b+c \cos ax} \right| + C, \quad b^2 < c^2$
76. $\int \frac{dx}{1+\cos ax} = \frac{1}{a} \tan \frac{ax}{2} + C$
77. $\int \frac{dx}{1-\cos ax} = -\frac{1}{a} \cot \frac{ax}{2} + C$
78. $\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$
79. $\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$
80. $\int x^n \sin ax \, dx = -\frac{x^n}{a} \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, dx$
81. $\int x^n \cos ax \, dx = \frac{x^n}{a} \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \, dx$
82. $\int \tan ax \, dx = \frac{1}{a} \ln |\sec ax| + C$
83. $\int \cot ax \, dx = \frac{1}{a} \ln |\sin ax| + C$
84. $\int \tan^2 ax \, dx = \frac{1}{a} \tan ax - x + C$
85. $\int \cot^2 ax \, dx = -\frac{1}{a} \cot ax - x + C$
86. $\int \tan^n ax \, dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int \tan^{n-2} ax \, dx, \quad n \neq 1$
87. $\int \cot^n ax \, dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int \cot^{n-2} ax \, dx, \quad n \neq 1$
88. $\int \sec ax \, dx = \frac{1}{a} \ln |\sec ax + \tan ax| + C$
89. $\int \csc ax \, dx = -\frac{1}{a} \ln |\csc ax + \cot ax| + C$
90. $\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C$
91. $\int \csc^2 ax \, dx = -\frac{1}{a} \cot ax + C$
92. $\int \sec^n ax \, dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2} ax \, dx, \quad n \neq 1$
93. $\int \csc^n ax \, dx = -\frac{\csc^{n-2} ax \cot ax}{a(n-1)} + \frac{n-2}{n-1} \int \csc^{n-2} ax \, dx, \quad n \neq 1$
94. $\int \sec^n ax \tan ax \, dx = \frac{\sec^n ax}{na} + C, \quad n \neq 0$
95. $\int \csc^n ax \cot ax \, dx = -\frac{\csc^n ax}{na} + C, \quad n \neq 0$

96. $\int \sin^{-1} ax \, dx = x \sin^{-1} ax + \frac{1}{a} \sqrt{1 - a^2 x^2} + C$
97. $\int \cos^{-1} ax \, dx = x \cos^{-1} ax - \frac{1}{a} \sqrt{1 - a^2 x^2} + C$
98. $\int \tan^{-1} ax \, dx = x \tan^{-1} ax - \frac{1}{2a} \ln(1 + a^2 x^2) + C$
99. $\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1 - a^2 x^2}}, \quad n \neq -1$
100. $\int x^n \cos^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \cos^{-1} ax + \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1 - a^2 x^2}}, \quad n \neq -1$
101. $\int x^n \tan^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \tan^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{1 + a^2 x^2}, \quad n \neq -1$
102. $\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$
103. $\int b^{ax} \, dx = \frac{1}{a} \frac{b^{ax}}{\ln b} + C, \quad b > 0, b \neq 1$
104. $\int x e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1) + C$
105. $\int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx$
106. $\int x^n b^{ax} \, dx = \frac{x^n b^{ax}}{a \ln b} - \frac{n}{a \ln b} \int x^{n-1} b^{ax} \, dx, \quad b > 0, b \neq 1$
107. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$
108. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$
109. $\int \ln ax \, dx = x \ln ax - x + C$
110. $\int x^n (\ln ax)^m \, dx = \frac{x^{n+1} (\ln ax)^m}{n+1} - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} \, dx, \quad n \neq -1$
111. $\int x^{-1} (\ln ax)^m \, dx = \frac{(\ln ax)^{m+1}}{m+1} + C, \quad m \neq -1$
112. $\int \frac{dx}{x \ln ax} = \ln |\ln ax| + C$
113. $\int \sinh ax \, dx = \frac{1}{a} \cosh ax + C$
114. $\int \cosh ax \, dx = \frac{1}{a} \sinh ax + C$
115. $\int \sinh^2 ax \, dx = \frac{\sinh 2ax}{4a} - \frac{x}{2} + C$
116. $\int \cosh^2 ax \, dx = \frac{\sinh 2ax}{4a} + \frac{x}{2} + C$
117. $\int \sinh^n ax \, dx = \frac{\sinh^{n-1} ax \cosh ax}{na} - \frac{n-1}{n} \int \sinh^{n-2} ax \, dx, \quad n \neq 0$
118. $\int \cosh^n ax \, dx = \frac{\cosh^{n-1} ax \sinh ax}{na} + \frac{n-1}{n} \int \cosh^{n-2} ax \, dx, \quad n \neq 0$
119. $\int x \sinh ax \, dx = \frac{x}{a} \cosh ax - \frac{1}{a^2} \sinh ax + C$
120. $\int x \cosh ax \, dx = \frac{x}{a} \sinh ax - \frac{1}{a^2} \cosh ax + C$
121. $\int x^n \sinh ax \, dx = \frac{x^n}{a} \cosh ax - \frac{n}{a} \int x^{n-1} \cosh ax \, dx$
122. $\int x^n \cosh ax \, dx = \frac{x^n}{a} \sinh ax - \frac{n}{a} \int x^{n-1} \sinh ax \, dx$
123. $\int \tanh ax \, dx = \frac{1}{a} \ln(\cosh ax) + C$
124. $\int \coth ax \, dx = \frac{1}{a} \ln |\sinh ax| + C$

$$125. \int \tanh^2 ax \, dx = x - \frac{1}{a} \tanh ax + C$$

$$126. \int \coth^2 ax \, dx = x - \frac{1}{a} \coth ax + C$$

$$127. \int \tanh^n ax \, dx = -\frac{\tanh^{n-1} ax}{(n-1)a} + \int \tanh^{n-2} ax \, dx, \quad n \neq 1$$

$$128. \int \coth^n ax \, dx = -\frac{\coth^{n-1} ax}{(n-1)a} + \int \coth^{n-2} ax \, dx, \quad n \neq 1$$

$$129. \int \operatorname{sech} ax \, dx = \frac{1}{a} \sin^{-1}(\tanh ax) + C$$

$$130. \int \operatorname{csch} ax \, dx = \frac{1}{a} \ln \left| \tanh \frac{ax}{2} \right| + C$$

$$131. \int \operatorname{sech}^2 ax \, dx = \frac{1}{a} \tanh ax + C$$

$$132. \int \operatorname{csch}^2 ax \, dx = -\frac{1}{a} \coth ax + C$$

$$133. \int \operatorname{sech}^n ax \, dx = \frac{\operatorname{sech}^{n-2} ax \tanh ax}{(n-1)a} + \frac{n-2}{n-1} \int \operatorname{sech}^{n-2} ax \, dx, \quad n \neq 1$$

$$134. \int \operatorname{csch}^n ax \, dx = -\frac{\operatorname{csch}^{n-2} ax \coth ax}{(n-1)a} - \frac{n-2}{n-1} \int \operatorname{csch}^{n-2} ax \, dx, \quad n \neq 1$$

$$135. \int \operatorname{sech}^n ax \tanh ax \, dx = -\frac{\operatorname{sech}^n ax}{na} + C, \quad n \neq 0$$

$$136. \int \operatorname{csch}^n ax \coth ax \, dx = -\frac{\operatorname{csch}^n ax}{na} + C, \quad n \neq 0$$

$$137. \int e^{ax} \sinh bx \, dx = \frac{e^{ax}}{2} \left[\frac{e^{bx}}{a+b} - \frac{e^{-bx}}{a-b} \right] + C, \quad a^2 \neq b^2$$

$$138. \int e^{ax} \cosh bx \, dx = \frac{e^{ax}}{2} \left[\frac{e^{bx}}{a+b} + \frac{e^{-bx}}{a-b} \right] + C, \quad a^2 \neq b^2$$

$$139. \int_0^{\infty} x^{n-1} e^{-x} \, dx = \Gamma(n) = (n-1)!, \quad n > 0$$

$$140. \int_0^{\infty} e^{-ax^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}, \quad a > 0$$

$$141. \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \cdot \frac{\pi}{2}, & \text{if } n \text{ is an even integer } \geq 2 \\ \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n}, & \text{if } n \text{ is an odd integer } \geq 3 \end{cases}$$

Taylor Series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-x)^n + \dots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$$

$$\ln \frac{1+x}{1-x} = 2 \tanh^{-1} x = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n+1}}{2n+1} + \dots \right) = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1$$

Binomial Series

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \dots + \frac{m(m-1)(m-2)\dots(m-k+1)x^k}{k!} + \dots$$

$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \quad |x| < 1,$$

where

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!}, \quad \binom{m}{k} = \frac{m(m-1)\dots(m-k+1)}{k!} \quad \text{for } k \geq 3.$$

LIMITS

General Laws

If L , M , c , and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

Sum Rule: $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$

Difference Rule: $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$

Product Rule: $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$

Constant Multiple Rule: $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$

Quotient Rule: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

The Sandwich Theorem

If $g(x) \leq f(x) \leq h(x)$ in an open interval containing c , except possibly at $x = c$, and if

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L,$$

then $\lim_{x \rightarrow c} f(x) = L$.

Inequalities

If $f(x) \leq g(x)$ in an open interval containing c , except possibly at $x = c$, and both limits exist, then

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x).$$

Continuity

If g is continuous at L and $\lim_{x \rightarrow c} f(x) = L$, then

$$\lim_{x \rightarrow c} g(f(x)) = g(L).$$

Specific Formulas

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

If $f(x)$ is continuous at $x = c$, then

$$\lim_{x \rightarrow c} f(x) = f(c).$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

L'Hôpital's Rule

If $f(a) = g(a) = 0$, both f' and g' exist in an open interval I containing a , and $g'(x) \neq 0$ on I if $x \neq a$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right side exists.

General Formulas

Zero: $\int_a^a f(x) dx = 0$

Order of Integration: $\int_b^a f(x) dx = -\int_a^b f(x) dx$

Constant Multiples: $\int_a^b kf(x) dx = k \int_a^b f(x) dx$ (Any number k)

$\int_a^b -f(x) dx = -\int_a^b f(x) dx$ ($k = -1$)

Sums and Differences: $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

Additivity: $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

Max-Min Inequality: If $\max f$ and $\min f$ are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b - a) \leq \int_a^b f(x) dx \leq \max f \cdot (b - a).$$

Domination: $f(x) \geq g(x)$ on $[a, b]$ implies $\int_a^b f(x) dx \geq \int_a^b g(x) dx$

$f(x) \geq 0$ on $[a, b]$ implies $\int_a^b f(x) dx \geq 0$

The Fundamental Theorem of Calculus

Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

Part 2 If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Substitution in Definite Integrals

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Integration by Parts

$$\int_a^b f(x)g'(x) dx = f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x) dx$$

VECTOR OPERATOR FORMULAS (CARTESIAN FORM)

Formulas for Grad, Div, Curl, and the Laplacian

	<p>Cartesian (x, y, z) \mathbf{i}, \mathbf{j}, and \mathbf{k} are unit vectors in the directions of increasing x, y, and z. M, N, and P are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.</p>
Gradient	$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$
Divergence	$\nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Vector Triple Products

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Vector Identities

In the identities here, f and g are differentiable scalar functions, \mathbf{F} , \mathbf{F}_1 , and \mathbf{F}_2 are differentiable vector fields, and a and b are real constants.

$$\nabla \times (\nabla f) = \mathbf{0}$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla \cdot (g\mathbf{F}) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$$

$$\nabla \times (g\mathbf{F}) = g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$$

$$\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$$

$$\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$$

$$\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 +$$

$$\mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$$

The Fundamental Theorem of Line Integrals

- Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D .

- If the integral is independent of the path from A to B , its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem and Its Generalization to Three Dimensions

Normal form of Green's Theorem: $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \nabla \cdot \mathbf{F} \, dA$

Divergence Theorem: $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV$

Tangential form of Green's Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA$

Stokes' Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$

$$\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$$

$$\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla)\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

$$(\nabla \times \mathbf{F}) \times \mathbf{F} = (\mathbf{F} \cdot \nabla)\mathbf{F} - \frac{1}{2}\nabla(\mathbf{F} \cdot \mathbf{F})$$