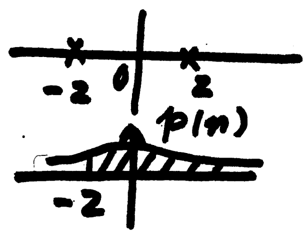
Chang, Shih-Chun "Homework Solutions No. 04". Lecture Notes.
ECE 630. Spring 2011. 1-7

HWS10030211.pdf

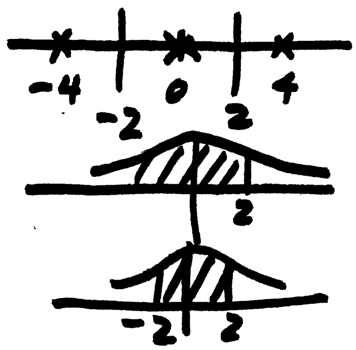
1 **4.1** (a) $n \sim N(0,\sigma^2)$, $P[c|s=2]=P[c|s=-2]$,



$P[c|s=2] = P[r>0 \,|\, s=2]$

$= P[n+2>0] = P[n>-2]$

$= 1-Q(2/\sigma) \rightarrow P[e] = \underline{\underline{Q(2/\sigma) = 0.01}}.$



$P[c|s=-4] = P[c|s=4]$; $P[c|s=4]$

$= P[n-4<-2] = P[n<2]$

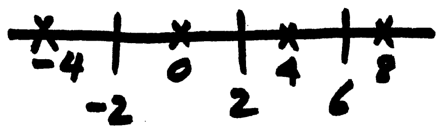
$= 1-Q(2/\sigma)$

$P[c|s=0] = P[|n|<2] = 1-2Q(\frac{2}{\sigma})$

$\rightarrow P[c] = \frac{2}{3}[1-Q(2/\sigma)] + \frac{1}{3}[1-2Q(\frac{2}{\sigma})]$

$= 1-(\frac{2}{3}+\frac{2}{3})Q(2/\sigma) = 1-\frac{4}{3}Q(2/\sigma)$

$\rightarrow P[e] = \frac{4}{3}Q(2/\sigma) = \underline{\underline{0.0133}}.$



$P[c|s=-4] = P[c|s=8]$

$= 1-Q(2/\sigma)$

$P[c|s=0] = P[c|s=4]$

$= 1-2Q(\frac{2}{\sigma})$

$\rightarrow P[c] = \frac{1}{2}[1-Q(2/\sigma)] + \frac{1}{2}[1-2Q(2/\sigma)] = 1-\frac{3}{2}Q(\frac{2}{\sigma})$

$\rightarrow P[e] = \frac{3}{2}Q(2/\sigma) = \underline{\underline{0.015}}.$

(b) $n \sim N(1, \sigma^2)$, $s=-2$, $Y=-2+n \to \bar{Y}=-2+1=-1$

$$\sigma_Y^2 = \overline{(Y+1)^2} = \overline{(-2+1+n)^2} = \overline{(n-1)^2} = \sigma^2 \; ; \; s=2 \to Y=2+n \to \bar{Y}=3, \quad \overline{(Y-3)^2} = \overline{(n-1)^2} = \sigma^2$$

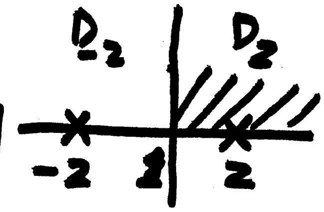
$$\to p(Y \mid s=-2) \underset{+2}{\overset{-2}{\gtrless}} p(Y \mid s=2)$$

$$\to e^{-\frac{(Y+1)^2}{2\sigma^2}} \underset{2}{\overset{-2}{\gtrless}} e^{-\frac{(Y-3)^2}{2\sigma^2}}$$

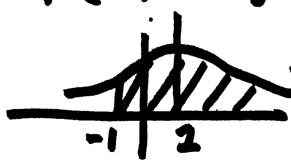
$$\to -(Y+1)^2 \underset{2}{\overset{-2}{\gtrless}} -(Y-3)^2$$

$$\to -2Y-1 \underset{2}{\overset{-2}{\gtrless}} +6Y-9$$

$$\to 8 \underset{2}{\overset{-2}{\gtrless}} 8Y \to \boxed{1 \underset{2}{\overset{-2}{\gtrless}} Y}$$



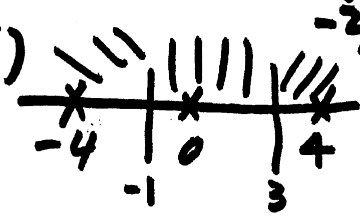
$$P[c \mid s=2] = P[n+2 > 1] = P[n > -1]$$



$$= \int_{-1}^{\infty} \frac{e^{-\frac{(n-1)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dn = \underline{\underline{Q(2/\sigma) = 0.01}}$$

$$\left(= \int_{-2/\sigma}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx, \quad x = \frac{n-1}{\sigma} \right)$$

(i)



$$P[c \mid s=-4] = P[n-4 < -1]$$

$$= P[n < 3] = P[n'$$

$$= \int_{-\infty}^{3} \frac{e^{-\frac{(n-1)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dn = \int_{-\infty}^{\frac{2}{\sigma}} \frac{e^{-n'^2/2}}{\sqrt{2\pi}}\, dn'$$

$$= 1 - Q(2/\sigma)$$

$$n' = \frac{n-1}{\sigma}$$

$$P[c \mid s=4] = P[n+4>3] = P[n>-1]$$

$$= \int_{-1}^{\infty} \frac{e^{-(n-1)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dn = \int_{-2/\sigma}^{\infty} \frac{e^{-n'^2/2}}{\sqrt{2\pi}}\,dn' = 1-Q(2/\sigma)$$

$$P[c \mid s=0] = P[-1<n<3] = \int_{-1}^{3} \frac{e^{-(n-1)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dn$$

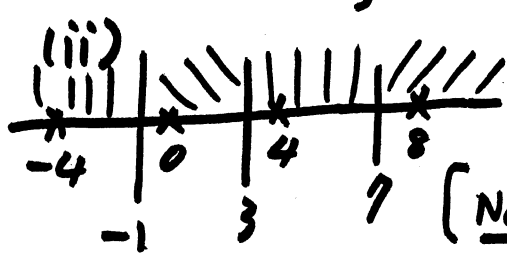
$$= \int_{-2/\sigma}^{2/\sigma} \frac{e^{-n'^2/2}}{\sqrt{2\pi}}\,dn' = 1-2Q(2/\sigma)$$

$$\rightarrow P[c] = \frac{1}{3}\left[2-2Q(2/\sigma)\right] + \frac{1}{3}\left[1-2Q(2/\sigma)\right]$$

$$= 1-\frac{4}{3}Q(2/\sigma)$$

$$\rightarrow P[e] = \frac{4}{3}Q(2/\sigma) = \underline{\underline{0.0133}} \quad \underline{(\text{NO Change})}$$

(iii)

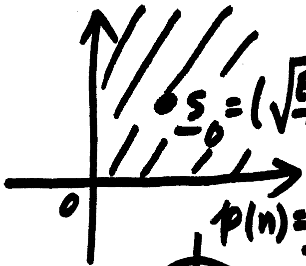


$P[e] = \underline{\underline{0.015}}$ $\underline{(\text{No change})}$,

[Note: Optimum Decision Regions had been shifted to the right by "1"!]

3 4.2 :

4

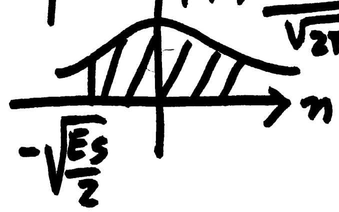


$$P[\text{correct} \mid \underline{s}_0]$$

$$\underline{s}_0 = \left(\sqrt{\tfrac{E_s}{2}}, \sqrt{\tfrac{E_s}{2}}\right) = P[\, r_1 > 0,\ r_2 > 0 \mid \underline{s}_0]$$

$$= P\left[n_1 + \sqrt{\tfrac{E_s}{2}} > 0\right] P\left[n_2 + \sqrt{\tfrac{E_s}{2}} > 0\right]$$

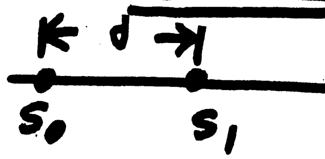
$$p(n) = \frac{e^{-n^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \qquad = \left[1 - Q\left(\sqrt{\tfrac{E_s}{N_0}}\right)\right]^2$$



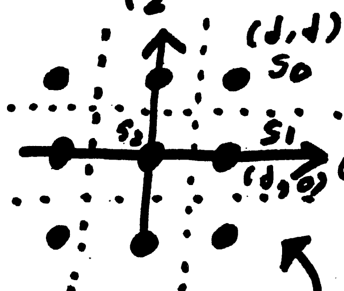
$$\rightarrow P[\text{error}] = 1 - \left[1 - Q\left(\sqrt{\tfrac{E_s}{N_0}}\right)\right]^2$$

$$= 2Q\left(\sqrt{\tfrac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\tfrac{E_s}{N_0}}\right)$$

3 4.3 :



$$P[e] = Q\left(\frac{d}{2\sigma}\right) \triangleq q$$



$$P[c \mid s_0] = P\left[n_1' + d > d/2\right]^2$$

$$= \left[1 - Q\left(\frac{d}{2\sigma}\right)\right]^2$$

$$P[c \mid s_1] = P\left[n_1' + d > d/2\right]$$

(Fig. P4.3 → Rotation & Translation)

$$\times P\left[|n_2'| < d/2\right]$$

$$= \left[1 - Q\left(\frac{d}{2\sigma}\right)\right]\left[1 - 2Q\left(\frac{d}{2\sigma}\right)\right]$$

$$P[c \mid s_2] = P\left[|n_1| < d/2\right]^2$$

$$= \left[1 - 2Q\left(\frac{d}{2\sigma}\right)\right]^2$$

$$\rightarrow P[\text{correct}] = \frac{4}{9}(1-q)^2 + \frac{4}{9}(1-q)(1-2q) + \frac{1}{9}(1-2q)^2$$

$$= 1 - \frac{8}{3}q + \frac{16}{9}q^2$$

$$\rightarrow P[\text{error}] = \frac{8}{3}q - \frac{16}{9}q^2$$

$\boxed{4}$ 4.6 (a) ~ (c): ⑤

(a) $P[m_0] = P[m_1] = 1/2$, $s_0 = -\sqrt{E_s}$, $s_1 = \sqrt{E_s}$.

Given $m = m_1$, $\begin{cases} r_1 = \sqrt{E_s} + n_1, \\ r_2 = n_1 + n_2, \end{cases}$

where $n_1, n_2 \sim IID(0, \sigma^2)$.

$$p(\underline{r} \mid m_1) \stackrel{\Delta}{=} p(r_1 \mid m_1)\, p(r_2 \mid r_1, m_1)$$

$$= \frac{e^{-\frac{(r_1 - \sqrt{E_s})^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \cdot \frac{e^{-\frac{(r_2 - r_1 + \sqrt{E_s})^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}},$$

(Given r_1, m_1, then $r_2 = r_1 - \sqrt{E_s} + n_2$.)

Similarly, $p(\underline{r} \mid m_0) = \dfrac{e^{-\frac{(r_1 + \sqrt{E_s})^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \dfrac{e^{-\frac{(r_2 - r_1 - \sqrt{E_s})^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$.

$$\rightarrow P[m_1]\, p(\underline{r} \mid m_1) \underset{D_0}{\overset{D_1}{\gtrless}} P[m_0]\, p(\underline{r} \mid m_0)$$

$$\rightarrow -(r_1 - \sqrt{E_s})^2 - (r_2 - r_1 + \sqrt{E_s})^2 \underset{D_0}{\overset{D_1}{\gtrless}} -(r_1 + \sqrt{E_s})^2 - (r_2 - r_1 - \sqrt{E_s})^2$$

$$\xrightarrow{\text{simplify}} 2r_1 - r_2 \underset{D_0}{\overset{D_1}{\gtrless}} \phi \rightarrow \boxed{r_1 - \tfrac{1}{2} r_2 \underset{D_0}{\overset{D_1}{\gtrless}} \phi}$$

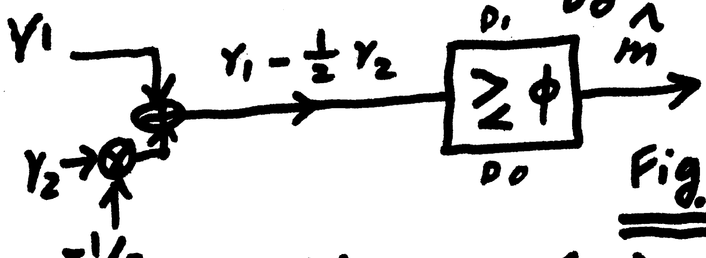


Fig P4.6(b)

(b) $\underline{\underline{a = -1/2}}$

(c) $\underline{\underline{\eta = \phi}}$.

(d) Given $m = m_1$, $p[\text{error} \mid m_1] = p[2r_1 - r_2 < 0 \mid m_1]$

$$r = 2r_1 - r_2 = 2(\sqrt{E_s} + n_1) - (n_1 + n_2)$$
$$= 2\sqrt{E_s} + n_1 - n_2 \sim N(2\sqrt{E_s}, 2\sigma^2)$$

$$\rightarrow p[\text{error} \mid m_1] = \int_{-\infty}^{0} \frac{e^{-\frac{(r - 2\sqrt{E_s})}{2(2\sigma^2)}}}{\sqrt{2\pi(2\sigma^2)}} \, dr$$

$$= Q\left(\frac{2\sqrt{E_s}}{\sqrt{2\sigma^2}}\right) = Q\left(\sqrt{\frac{2E_s}{\sigma^2}}\right)$$

Similarly, $p[\text{error} \mid m_0] = Q\left(\sqrt{\frac{2E_s}{\sigma^2}}\right)$

$$\rightarrow p[\text{error}] = Q\left(\sqrt{\frac{2E_s}{\sigma^2}}\right).$$

(e) $p[\text{error}] = Q\left(\sqrt{\frac{E_s}{\sigma^2}}\right)$, (if r_1 is used only)

Therefore the energy must be doubled, or 3dB increase in energy.

5 4.14: (Use MATLAB!)