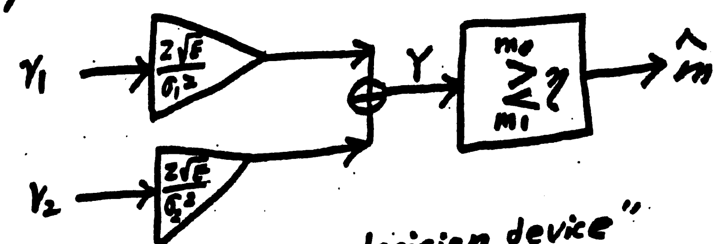
1 2.25: (a)

$$\begin{cases} p(\underline{y}\,|m_0) = \dfrac{e^{-\frac{(y_1-\sqrt{E})^2}{2\sigma_1^2}}}{\sqrt{2\pi\sigma_1^2}} \cdot \dfrac{e^{-\frac{(y_2-\sqrt{E})^2}{2\sigma_2^2}}}{\sqrt{2\pi\sigma_2^2}} \\ p(\underline{y}\,|m_1) = \dfrac{e^{-\frac{(y_1+\sqrt{E})^2}{2\sigma_1^2}}}{\sqrt{2\pi\sigma_1^2}} \cdot \dfrac{e^{-\frac{(y_2+\sqrt{E})^2}{2\sigma_2^2}}}{\sqrt{2\pi\sigma_2^2}} \end{cases}$$

MAP: $P[m_0]\, p(\underline{y}\,|m_0) \underset{m_1}{\overset{m_0}{\gtrless}} P[m_1]\, p(\underline{y}\,|m_1)$

Simplify: $\left(\frac{2\sqrt{E}}{\sigma_1^2}\right) y_1 + \left(\frac{2\sqrt{E}}{\sigma_2^2}\right) y_2 \underset{m_1}{\overset{m_0}{\gtrless}} \ln \frac{P[m_1]}{P[m_0]} = \eta$

(i)



"optimum decision device"

(ii) $m = m_1$, $\begin{cases} y_1 = -\sqrt{E} + n_1 \\ y_2 = -\sqrt{E} + n_2 \end{cases}$

$$Y = \frac{2\sqrt{E}}{\sigma_1^2}(-\sqrt{E} + n_1) + \frac{2\sqrt{E}}{\sigma_2^2}(-\sqrt{E} + n_2)$$

$$= (-2E)\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) + 2\sqrt{E}\left(\frac{n_1}{\sigma_1^2} + \frac{n_2}{\sigma_2^2}\right)$$

$$\rightarrow \bar{Y} = (-2E)\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right), \quad \sigma_y^2 = \frac{4E}{\sigma_1^2} + \frac{4E}{\sigma_2^2}$$

$$P[\text{error}|m_1] = P[Y > \eta] = \int_\eta^\infty \frac{e^{-\frac{(y-\bar{y})^2}{2\sigma_y^2}}}{\sqrt{2\pi\sigma_y^2}}\, dy \quad (1)$$

$$m = m_0, \begin{cases} r_1 = \sqrt{E} + n_1 \\ r_2 = \sqrt{E} + n_2 \end{cases}$$

②

$$y = \frac{2\sqrt{E}}{\sigma_1^2}(\sqrt{E}+n_1) + \frac{2\sqrt{E}}{\sigma_2^2}(\sqrt{E}+n_2)$$

$$= (2E)\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right) + 2\sqrt{E}\left(\frac{n_1}{\sigma_1^2}+\frac{n_2}{\sigma_2^2}\right)$$

$$\rightarrow \bar{y} = (2E)\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right), \quad \sigma_y^2 = \frac{4E}{\sigma_1^2}+\frac{4E}{\sigma_2^2}$$

$$P[\text{error} \mid m_0] = P[y < \eta] = \int_{-\infty}^{\eta} \frac{e^{-(y-\bar{y})^2/2\sigma_y^2}}{\sqrt{2\pi\sigma_y^2}}\,dy \quad (2)$$

From (1) & (2) $\rightarrow$ $P[\text{error}] = P[\text{error} \mid m_0]\,P[m_0] + P[\text{error} \mid m_1]\,P[m_1]$.

(b)

$$\begin{cases} P[\text{error} \mid m_1] = Q\left(\dfrac{4E/\sigma^2}{\sqrt{8E/\sigma^2}}\right) = Q\left(\sqrt{\dfrac{2E}{\sigma^2}}\right), \\ P[\text{error} \mid m_0] = Q\left(\dfrac{4E/\sigma^2}{\sqrt{8E/\sigma^2}}\right) = Q\left(\sqrt{\dfrac{2E}{\sigma^2}}\right), \\ P[\text{error}] = Q\left(\sqrt{\dfrac{2E}{\sigma^2}}\right). \end{cases}$$

Based on r_1 only, $P[\text{error}] = Q\left(\sqrt{\frac{E}{\sigma^2}}\right) > Q\left(\sqrt{\frac{2E}{\sigma^2}}\right)$.

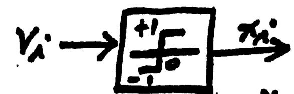
$\boxed{3}$ **2.39:**

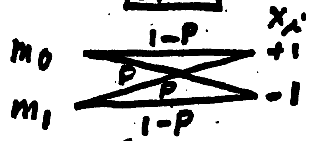
(b)

(i)

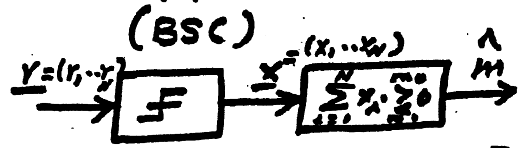
$m = m_0,\ r_i = \sqrt{E} + n_i,\quad P[r_i < 0 \mid m_0] = P[\sqrt{E} + n_i < 0] = P[n_i < -\sqrt{E}]$

$$= \int_{-\infty}^{-\sqrt{E}} \frac{e^{-n_i^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, dn_i = Q\left(\sqrt{E/\sigma^2}\right) = p$$





(BSC)



(ii) $P[\text{error} \mid m_1] = P[x > 0 \mid m_1]$ $\quad \begin{cases} x = \sum_{i=1}^{N} x_i, \\ P[x_i = 1 \mid m_1] = p, \\ P[x_i = 0 \mid m_1] = 1-p. \end{cases}$

$$\leq E\left[e^{\nu x}\right]$$

$$= E\left[e^{\nu(x_1 + \cdots + x_N)}\right]$$

$$= E\left[e^{\nu x_1}\right] E\left[e^{\nu x_2}\right] \cdots E\left[e^{\nu x_N}\right]$$

$$= \left[p e^{\nu} + (1-p) e^{-\nu}\right]^N$$

Chernoff Bound: $\frac{\partial}{\partial \nu}\left[p e^{\nu} + (1-p) e^{-\nu}\right]$

$$= p e^{\nu} - (1-p) e^{-\nu} = \phi \rightarrow e^{\nu} = \sqrt{\frac{1-p}{p}} > 1.$$

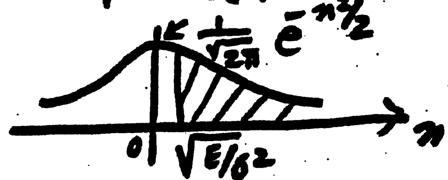
$$\rightarrow P[\text{error}] \leq \left[\sqrt{4p(1-p)}\right]^N$$

→ By symmetry,

$$P[\text{error}] \le e^{-N\left[-\ln\sqrt{4p(1-p)}\right]}$$

$$= e^{-N\left\{-\frac{1}{2}\ln[4p(1-p)]\right\}}$$

(iii) $p = Q\left(\sqrt{E/\sigma^2}\right)$



$$\rightarrow \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \times \sqrt{\frac{E}{\sigma^2}} \simeq p$$

$$\rightarrow 4p(1-p) \approx 1 - \frac{2E}{\pi\sigma^2}, \quad \left(\frac{E}{\sigma^2} \ll 1\right)$$

$$\rightarrow -\frac{1}{2}\ln[4p(1-p)] = \frac{1}{2} \times \frac{2E}{\pi\sigma^2} = \frac{E}{\pi\sigma^2}$$

$$\rightarrow P_{\text{Hard}}[\text{error}] \le e^{-\frac{NE}{\pi\sigma^2}} = e^{-\left(\frac{NE}{2\sigma^2}\right)\left(\frac{2}{\pi}\right)}$$

$$\rightarrow P_{\text{soft}}[\text{error}] \le \frac{1}{2} e^{-\left(\frac{NE}{2\sigma^2}\right)}$$

$10\log_{10}\left(\frac{2}{\pi}\right) = 2$ dB (degradation)

$\boxed{3}$ **3.11**: $X, Y \sim$ IID $N(0, 1)$ (5)

(a) $\overline{z(t)} = \underline{\underline{\phi}}$

-I

$$R_z(t_1, t_2) = E\left[z(t_1)\, z(t_2)\right]$$

$$= \underline{\underline{\cos\left[2\pi(t_1 - t_2)\right]}}$$

$\rightarrow$ $z(t)$ is a **stationary** Gaussian random process.

(a)-II

$$\underset{\sim}{\Lambda}_z = \begin{bmatrix} 1 & \cos 2\pi(t_1 - t_2) \\ \cos 2\pi(t_1 - t_2) & 1 \end{bmatrix}$$

$$\underset{\sim}{\Lambda}_z^{-1} = \frac{1}{\left[1 - \cos^2 2\pi(t_1 - t_2)\right]} \begin{bmatrix} 1 & -\cos 2\pi(t_1 - t_2) \\ -\cos 2\pi(t_1 - t_2) & 1 \end{bmatrix}$$

$$\rightarrow \quad = \underline{\underline{\frac{\begin{bmatrix} +1 & -\cos 2\pi(t_1 - t_2) \\ -\cos 2\pi(t_1 - t_2) & +1 \end{bmatrix}}{\sin^2 2\pi(t_1 - t_2)}}}$$

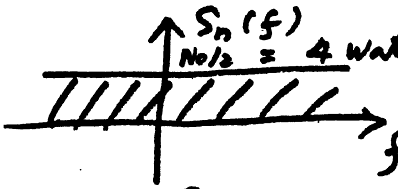
$$\& \quad |\underset{\sim}{\Lambda}_z| = \underline{\underline{\sin^2 2\pi(t_1 - t_2)}}$$

$$\rightarrow \quad p_z(\underline{z}) = \frac{e^{-\frac{1}{2}\underline{z}\,\underset{\sim}{\Lambda}_z^{-1}\,\underline{z}^{*}}}{(2\pi)\left|\underset{\sim}{\Lambda}_z\right|^{1/2}}$$

$$= \underline{\underline{\frac{e^{-\left\{\frac{z_1^2 - \left[2\cos 2\pi(t_1 - t_2)\right] z_1 z_2 + z_2^2}{\left[2\sin^2 2\pi(t_1 - t_2)\right]}\right\}}}{2\pi\,\left|\sin 2\pi(t_1 - t_2)\right|}}}$$

4 4.7 (6)

$$\begin{cases} \int_0^T x^2(t)\,dt = \int_0^T y^2(t)\,dt = 16 \text{ joules} = E_b \\ \int_0^T x(t)\,y(t)\,dt = 0 \end{cases} \rightarrow x(t) \text{ \& } y(t) \text{ are equal-energy orthogonal signals.}$$



$\rightarrow N_0 = 8$

(a) $\rho = -1 \rightarrow \overline{P[\text{error}]} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{32}{8}}\right) = \underline{\underline{Q(2)}}$

(b) $\rho = 0 \rightarrow \overline{P[\text{error}]} = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = \underline{\underline{Q(\sqrt{2})}}$

5 4.8 (a) $\frac{N_0}{2} = 0.15 \rightarrow N_0 = \underline{0.3}$, $\quad P[m_0] = P[m_1] = 1/2$,

(i) $\int_0^1 s_0^2(t)\,dt = 1$, $\quad \int_0^1 s_1^2(t)\,dt = \frac{1}{2}$, $\quad \int_0^1 s_0(t)s_1(t)\,dt = 0.$

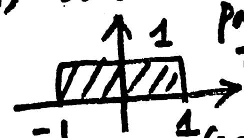
$E_b = \frac{1}{2} + \frac{1}{4} = \underline{\underline{\frac{3}{4}}} \rightarrow \overline{P[\text{error}]} = Q\left(\sqrt{\frac{(3/4)}{0.3}}\right) = \underline{\underline{Q(\sqrt{2.5})}} = \underline{\underline{Q(1.58)}}$

(ii) $\int_0^1 s_0^2(t)\,dt = 4$, $\quad \int_0^2 s_1^2(t)\,dt = 8$, $\quad \int_0^2 s_0(t)s_1(t)\,dt = 4$,

$E_b = \frac{4+8}{2} = \underline{\underline{6}}$, $\quad \rho = \frac{4}{6} = \underline{\underline{\frac{2}{3}}}$.

$\overline{P[\text{error}]} = Q\left(\sqrt{\frac{(1-\rho)E_b}{N_0}}\right) = Q\left(\sqrt{\frac{\frac{1}{3}\times 6}{0.3}}\right) = \underline{\underline{Q\left(\sqrt{\frac{20}{3}}\right)}} = \underline{\underline{Q(2.58)}}$

(iii) $S_0(f) \xrightarrow[\text{Theorem}]{\text{Parseval}} \int_{-\infty}^{\infty} s_0^2(t)\,dt = \int_{-1}^{1} S_0^2(f)\,df = 2$



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$\int_{-\infty}^{\infty} s_1^2(t)\,dt = \int_{-3/2}^{-1/2} S_1^2(f)\,df + \int_{1/2}^{3/2} S_1^2(f)\,df$

$= 2$

$\int_{-\infty}^{\infty} s_0(t)s_1(t)\,dt = 2 \cdot \int_{1/2}^{1} 1\,df = \underline{\underline{1}}$

$\rightarrow E_b = \frac{2+2}{2} = 2$, $\quad \rho = \frac{1}{2} = \underline{\underline{1/2}}$

$\overline{P[\text{error}]} = Q\left(\sqrt{\frac{(1-\rho)E_b}{N_0}}\right) = Q\left(\sqrt{\frac{\frac{1}{2}\times 2}{0.3}}\right) = \underline{\underline{Q\left(\sqrt{\frac{10}{3}}\right)}} = \underline{Q(1.82)}$

$$\rightarrow p_{\underline{z}}(\zeta_1,\zeta_2,\zeta_3) = p_{z_1,z_3}(\zeta_1,\zeta_3)\, p_{z_2}(\zeta_2)$$ 7

(z_2 is independent of ζ_1 and ζ_3)

$$\overline{z_2 z_3} = \overline{z_2 z_1} = 0$$

$$\overline{z_1 z_3} = -1 \longrightarrow \underline{\underline{z_1 = -z_3}}$$ ("degenerate" case)

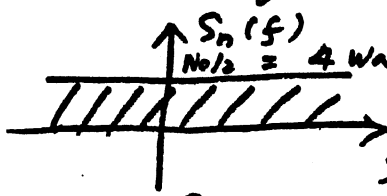
$$\rightarrow p_{z_1,z_3}(\zeta_1,\zeta_3) = p_{z_1}(\zeta_1)\,\delta(\zeta_1+\zeta_3)$$

$$p_{\underline{z}}(\zeta_1,\zeta_2,\zeta_3) = p_{z_1}(\zeta_1)\, p_{z_2}(\zeta_2)\,\delta(\zeta_1+\zeta_3)$$

$$\underline{\underline{= \frac{e^{-\left(\frac{\zeta_1^2+\zeta_2^2}{2}\right)}}{2\pi}\,\delta(\zeta_3+\zeta_1)}}$$

$\boxed{4}$ **4.7** $\begin{cases} \int_0^T x^2(t)\,dt = \int_0^T y^2(t)\,dt = 16 \text{ joules} = E_b \\ \int_0^T x(t)\,y(t)\,dt = 0 \end{cases} \longrightarrow x(t)\ \&\ y(t)$ are equal-energy orthogonal signals.

(8)



$\longrightarrow N_o = 8$

(a) $\rho = -1 \longrightarrow \overline{P[\text{error}]} = Q\left(\sqrt{\frac{2E_b}{N_o}}\right) = Q\left(\sqrt{\frac{32}{8}}\right) = \underline{\underline{Q(2)}}$

(b) $\rho = 0 \longrightarrow \overline{P[\text{error}]} = Q\left(\sqrt{\frac{E_b}{N_o}}\right) = \underline{\underline{Q(\sqrt{2})}}$

$\boxed{5}$ **4.8** (a) $\frac{N_o}{2} = 0.15 \longrightarrow N_o = \underline{0.3}$, $P[m_0] = P[m_1] = 1/2$,

(i) $\int_0^1 s_0^2(t)\,dt = 1$, $\int_0^1 s_1^2(t)\,dt = \frac{1}{2}$, $\int_0^1 s_0(t)s_1(t)\,dt = 0$.

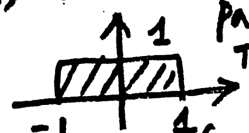
$E_b = \frac{1}{2} + \frac{1}{4} = \underline{\underline{\frac{3}{4}}} \longrightarrow \overline{P[\text{Error}]} = Q\left(\sqrt{\frac{(3/4)}{0.3}}\right) = \underline{\underline{Q(\sqrt{2.5})}} = \underline{\underline{Q(1.58)}}$

(ii) $\int_0^1 s_0^2(t)\,dt = 4$, $\int_0^2 s_1^2(t)\,dt = 8$, $\int_0^2 s_0(t)s_1(t)\,dt = 4$,

$E_b = \frac{4+8}{2} = \underline{\underline{6}}$, $\rho = \frac{4}{6} = \underline{\underline{\frac{2}{3}}}$.

$\overline{P[\text{error}]} = Q\left(\sqrt{\frac{(1-\rho)E_b}{N_o}}\right) = Q\left(\sqrt{\frac{\frac{1}{3}\times 6}{0.3}}\right) = \underline{\underline{Q\left(\sqrt{\frac{20}{3}}\right)}} = \underline{\underline{Q(2.58)}}$

(iii) $S_0(f) \xrightarrow[\text{Theorem}]{\text{Parseval}} \int_{-\infty}^{\infty} s_0^2(t)\,dt = \int_{-1}^{1} S_0^2(f)\,df = 2$



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$\int_{-\infty}^{\infty} s_1^2(t)\,dt = \int_{-3/2}^{-1/2} S_1^2(f)\,df + \int_{1/2}^{3/2} S_1^2(f)\,df$

$= 2$

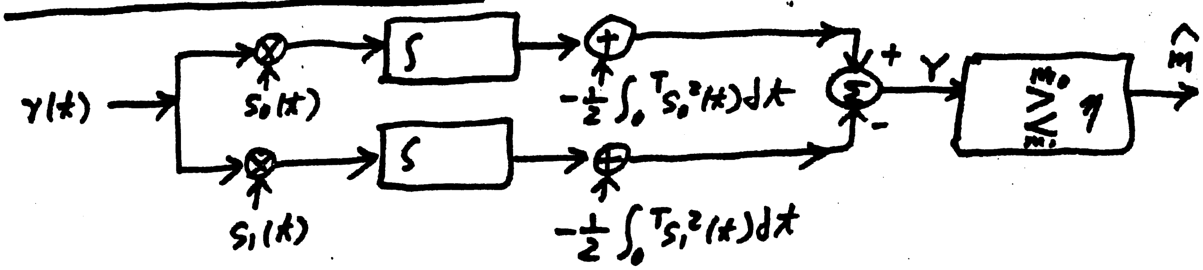
$\int_{-\infty}^{\infty} s_0(t)s_1(t)\,dt = 2 \cdot \int_{\frac{1}{2}}^{1} 1\,df = \underline{\underline{1}}$

$\rightarrow E_b = \frac{2+2}{2} = 2$, $\rho = \frac{1}{2} = \underline{\underline{1/2}}$

$\overline{P[\text{error}]} = Q\left(\sqrt{\frac{(1-\rho)E_b}{N_o}}\right) = Q\left(\sqrt{\frac{\frac{1}{2}\times 2}{0.3}}\right) = \underline{\underline{Q\left(\sqrt{\frac{10}{3}}\right)}} = \underline{\underline{Q(1.82)}}$

4.8 (b) : $P[m_0] = 1/4, \quad P[m_1] = 3/4$

new optimal receiver



where $\eta = \frac{N_0}{2} \log \frac{P[m_1]}{P[m_0]}$, $Y \triangleq \int_0^T r(t)\left[s_0(t) - s_1(t)\right] dt - \frac{1}{2}\int_0^T \left[s_0^2(t) - s_1^2(t)\right] dt$

$= 0.15 \log 3$

$$P[Y > \eta \mid m = m_1] \triangleq P[\text{error} \mid m_1]$$

$$= \int_\eta^\infty \frac{e^{-(y + E_b(1-\rho))^2 / 2 N_0 E_b(1-\rho)}}{\sqrt{2\pi N_0 E_b(1-\rho)}}\, dy$$

$$= Q\left(\frac{\eta + E_b(1-\rho)}{\sqrt{N_0 E_b(1-\rho)}}\right)$$

$$P[Y < \eta \mid m = m_0] = \int_{-\infty}^{\eta} \frac{e^{-(y - E_b(1-\rho))^2 / 2 N_0 (1-\rho)}}{\sqrt{2\pi N_0 E_b(1-\rho)}}\, dy$$

$$= Q\left(\frac{E_b(1-\rho) - \eta}{\sqrt{N_0 E_b(1-\rho)}}\right)$$

$$\overline{P[\text{error}]} = P[m_0]\, Q\left(\frac{E_b(1-\rho) - \eta}{\sqrt{N_0 E_b(1-\rho)}}\right) + P[m_1]\, Q\left(\frac{E_b(1-\rho) + \eta}{\sqrt{N_0 E_b(1-\rho)}}\right)$$

(i) $\overline{P(\text{error})} = \frac{1}{4} Q(1.23) + \frac{3}{4} Q(1.92)$

(ii) $\overline{P(\text{error})} = \frac{1}{4} Q(2.36) + \frac{3}{4} Q(2.79)$

(iii) $\overline{P(\text{error})} = \frac{1}{4} Q(1.52) + \frac{3}{4} Q(2.12)$